Multiscale perturbation of nonlinear partial differential equations and integrability

Antonio Degasperis
Sapienza, Universita' di Roma, Rome, Italy

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OUTLINE

- introduction
- formal setting
- multiscale equations
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- derivation of the DP equation

anharmonic oscillator : $\ddot{q} + V'(q) = 0$

$$V(q) = \frac{1}{2}\omega_0^2 \frac{q^2}{1 + q^2/L_1^2 + q^3/L_2^3}$$

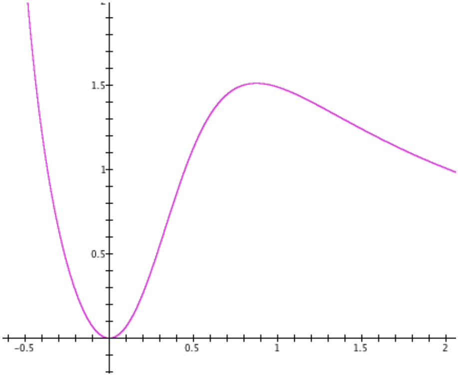

Figure: $V(q), \omega_0^2 = 14,\ L_1 = 1.195,\ L_2 = 0.693$

INTRODUCTION 2

periodic solution : $q(t)$, $q(0) = \epsilon$, $\dot{q}(0) = 0$
condition for existence : $|\epsilon| < 2^{1/3} L_2$

motion frequency : $\omega = \omega(\epsilon) = \omega_0 + \omega_1 \epsilon + \omega_2 \epsilon^2 + O(\epsilon^3)$

problem : find ω_n , for $n > 0$

solution : $\omega_1 = 0$, $\omega_2 = -\frac{3\omega_0}{4L_1^2}$, $\cdots$

INTRODUCTION 3

Poincare'-Lindstedt method :

$$q(t) = f(\theta, \epsilon) \;\; , \;\; \theta = \omega(\epsilon) t$$

$$\ddot{q} + \omega_0^2 q = c_2 q^2 + c_3 q^3 + \cdots \;\; \rightarrow \;\; \omega^2(\epsilon) f'' + \omega_0^2 f = c_2 f^2 + c_3 f^3 + \cdots$$

$$\begin{cases} f(\theta, \epsilon) & = & \epsilon f_1(\theta) + \epsilon^2 f_2(\theta) + \cdots \\ \omega(\epsilon) & = & \omega_0 + \epsilon\omega_1 + \epsilon^2 \omega_2 + \cdots \end{cases}$$

$$\begin{cases} f_n'' + f_n = F_n \; , \; f_1(0) = 1 \; , \; f_1'(0) = 0 \; , \; f_n(0) = 0 \; , \; f_n'(0) = 0 \; , \; n > 1 \\ F_1 = 0 \; , \; F_n = \sum_{\alpha=-n}^{n} F_n^{\alpha} e^{i\alpha\theta} \; , \; n > 1 \end{cases}$$

KILLING SECULAR TERMS : $F_n^{(1)} = F_n^{(-1)} = 0$ for $n = 2, 3, \cdots$

INTRODUCTION 4

Lindstedt series

$$q(t) = \sum_{n=1} \sum_{\alpha=-n}^{n} f_n^\alpha \epsilon^n e^{i\alpha(\omega_0 t + \epsilon\omega_1 t + \epsilon^2 \omega_2 t + \cdots)}$$

$$q(t) = \sum_{n=1} \sum_{\alpha=-n}^{n} f_n^\alpha \epsilon^n e^{i\alpha(\omega_0 t + \omega_1 t_1 + \omega_2 t_2 + \cdots)}$$

$t =$ fast time ; $t_n = \epsilon^n t$ slow times for $n = 1\,,\,2\,,\,3\cdots$

$$q(t) = \sum_{n=1} \sum_{\alpha=-n}^{n} \epsilon^n q_n^\alpha (t_1, t_2, \cdots) e^{i\alpha\omega_0 t}$$

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \cdots$$

1. nonlinearity generates $2n+1$ higher harmonics at each n
2. the approximate solution $q(t)$ is a double expansion

WAVE EQUATIONS 1

$$Du = F[u, u_x, u_{xx}, \ldots] \ , \quad u^* = u = u(x,t)$$

$$D = \partial/\partial t + i\omega(-i\partial/\partial x)$$

$$\omega(k) = \sum_{m=0} a_{2m+1} k^{2m+1}$$

example : $\omega(k) = a_1 k + a_3 k^3$

$$u_t + a_1 u_x - a_3 u_{xxx} = -a_3 u_x[\alpha \sinh u + \beta(\cosh u - 1) + u_x^2/8]$$

WAVE EQUATIONS 2

linear equation :

$$F = 0 \ , \ u(x,t) = \int_{-\infty}^{+\infty} dq U(q) \exp\{i[xq - t\omega(q)]\} + c.c.$$

$q = k + \eta\Delta k \ , \ -1 < \eta < 1 \ , \ U(q) = U(k + \eta\Delta k) = A(\eta)$
small parameter : $\epsilon = \Delta k / k$, carrier wave : $E(x,t) \equiv \exp[i(kx - \omega t)]$

$$u(x,t) = \epsilon E(x,t) u^{(1)}(\xi, t_1, t_2, \dots) + c.c.$$

$$\xi \equiv \epsilon x \ , \ t_n \equiv \epsilon^n t$$

$$u^{(1)}(\xi, t_1, t_2, \dots) = k \int_{-\infty}^{+\infty} d\eta A(\eta) \exp[i(k\eta\xi - k\omega_1\eta t_1 - k^2\omega_2\eta^2 t_2 - \dots)]$$

$$\partial_{t_n} u^{(1)} = (-i)^{n+1} \omega_n \partial_\xi^n u^{(1)} \ , \ n = 1, 2, \dots \ ,$$

$$[\partial_{t_n}, \partial_{t_m}] = 0$$

WAVE EQUATIONS 3

nonlinear equation : $Du = F[u, u_x, u_{xx}, \dots]$

$$u(x,t) = \sum_{\alpha=-\infty}^{+\infty} u^{(\alpha)}(\xi, t_1, t_2, \dots) E^{\alpha}(x,t) \ , \quad u^{(\alpha)*} = u^{(-\alpha)}$$

$$\partial_x \to \partial_x + \epsilon \partial_\xi \ , \ \partial_t \to \partial_t + \epsilon \partial_{t_1} + \epsilon^2 \partial_{t_2} + \dots$$

$$D = \partial/\partial t + i\omega(-i\partial/\partial x) \quad , \quad D[u^{(\alpha)} E^{\alpha}] = E^{\alpha} D^{(\alpha)} u^{(\alpha)}$$

$$F[u, u_x, u_{xx}, \dots] = \sum_{\alpha=-\infty}^{+\infty} F^{(\alpha)}[u^{(\beta)}, u_{\xi}^{(\beta)}, u_{\xi\xi}^{(\beta)}, \dots] E^{\alpha} \quad , \quad F^{(\alpha)*} = F^{(-\alpha)}$$

$$D^{(\alpha)} u^{(\alpha)} = F^{(\alpha)}$$

Remark 1 : if $F(-u) = -F(u)$ we may assume $u^{(2\alpha)} = 0$

WAVE EQUATIONS 4

$$D^{(\alpha)} = D_0^{(\alpha)} + \epsilon D_1^{(\alpha)} + \epsilon^2 D_2^{(\alpha)} + \ldots$$

$$D^{(\alpha)} = -i\alpha\omega(k) + \epsilon\partial_{t_1} + \epsilon^2\partial_{t_2} + \ldots + i\omega(\alpha k - i\epsilon\partial_\xi)$$

Remark 2 : if $\omega(k)$ is polynomial of degree N then the number of slow times is N and $D^{(\alpha)}$ is of order N

$$D_0^{(\alpha)} = i[\omega(\alpha k) - \alpha\omega(k)]$$

Remark 3 : $\begin{cases} \text{slave harmonics} \quad D_0^{(\alpha)} \neq 0 \\ \text{resonant harmonics} \quad D_0^{(\alpha)} = 0 \end{cases}$

Assumption : only the harmonics $\alpha = \pm 1$ are resonant

Remark 4 : $u^{(\alpha)} = O(\epsilon^\alpha)$, $u^{(\alpha)} = \sum_{n=\alpha} \epsilon^n u^{(\alpha)}(n)$

Remark 5 : all $u^{(\alpha)}(n)$ for $n \geq 2$ are differential polynomials of $u^{(1)}(n)$

WAVE EQUATIONS 5

new notation **ATTENTION** :

$$u^{(1)} \equiv u = \epsilon u(1) + \epsilon^2 u(2) + \dots \ , \ u^{(-1)} = u^* \ , \ u_j(n) \equiv \partial_\xi^j u(n)$$

$$D^{(1)} \equiv D = \epsilon D_1 + \epsilon^2 D_2 + \dots \ , \ D_n = \partial_{t_n} - (-i)^{n+1} \omega_n(k) \partial_\xi^n$$

$$F^{(1)} \equiv F = \epsilon^3 F_3 + \epsilon^4 F_4 + \dots$$

$\mathcal{P}_n \equiv$ set of nonlinear differential polynomials of order n and gauge1 of $u(m)$

$$\text{order of } \{u_j(n)\} = \text{order of } \{u_j^*(n)\} = j + n \ , \ F_n \in \mathcal{P}_n$$

$\dim\{\mathcal{P}_3\} = 1$, basis of $\{\mathcal{P}_3\} = \{|u(1)|^2 u(1)\}$, $\dim\{\mathcal{P}_4\} = 4$,
basis of $\{\mathcal{P}_4\} = \{|u(1)|^2 u(2) \ , \ u(1)^2 u^*(2) \ , \ |u(1)|^2 u_1(1) \ , \ u(1)^2 u_1^*(1)\}$
$\dim\{\mathcal{P}_5\} = 14$, $\dim\{\mathcal{P}_6\} = 36$,

$$\mathcal{P}_n(j) \equiv \text{ subspace of } \mathcal{P}_n \text{ depending on } u(m) \ , \ 1 \le m \le j$$

$$\mathcal{P}_n(n-2) = \mathcal{P}_n$$

SECULARITIES 1

resonant harmonic $\alpha = 1$: $Du = F$

$$D_1 u(n) + D_2 u(n-1) + \ldots + D_n u(1) = F_{n+1} \ , \quad n = 1, 2, 3, \cdots \ ,$$

$$n = 1 \ : \ D_1 u(1) = (\partial_{t_1} + \omega_k \partial_\xi) u(1) = 0 \ , \quad u(1) = u(1)(\xi - \omega_k t_1, t_2, t_3, \cdots)$$

$$n = 2 \ : \ D_1 u(2) = -D_2 u(1) + F_3 \ ,$$

Remark : $D_1 u(2) = 0 \, , \ u(2) = u(2)(\xi - \omega_k t_1, t_2, t_3, \cdots)$
proof : $D_1 D_2 u(1) = 0 \, , \ D(1) F_3 = 0 \ , \quad \to D_2 u(1) - F_3 = 0$

$$F_3 = -2i\omega_2 c |u(1)|^2 u(1)$$

NLS : $\partial_{t_2} u(1) = i\omega_2 (\partial_\xi^2 u(1) - 2c|u(1)|^2 u(1)) \ , \ c = c^*$
Remark : $D_1 u(n) = 0$ proof : by recursion

SECULARITIES 2

triangular system of PDEs

$$D_2 u(n-1) + D_3 u(n-2) + \ldots + D_n u(1) = F_{n+1} \ , \quad n = 2, 3, \cdots$$

$n = 2 \rightarrow$ NLS , $n = 3 \rightarrow \quad D_2 u(2) = -D_3 u(1) + F_4$
Remark : $D_3 u(1)$ is secular
eliminating secularity : $D_3 u(1) = -6\omega_3 c |u(1)|^2 u_\xi(1) \in \mathcal{P}_4(1)$, cmKdV
by recursion : $D_n u(1) = (-i)^{n+1} \omega_n c V_{n+1}, \ , \quad V_n \in \mathcal{P}_n(1) \quad n = 1, 2, \ldots$

$$V_5 = 2\,(3q_1 + 3cq_0^2 - q_{0\xi\xi})u(1) - 6(q_0 u_1(1))_\xi \ \in \mathcal{P}_5(1)$$

$$V_6 = 10\,(q_1 + 3cq_0^2 - q_{0\xi\xi})u_1(1) - 6(q_0 u_2(1))_\xi \ \in \mathcal{P}_6(1)$$

$$q_j = |u_j(1)|^2 \ , \quad j = 0, 1, 2, \ldots$$

$$D_2 u(n-1) + D_3 u(n-2) + \ldots + D_{n-1} u(2) = H_{n+1} \ , \ n \geq 3$$

$$H_n = F_n + (-i)^{n-1} \omega_{n-1} c V_n$$

INTEGRABILITY 1

integrability of NLS $\rightarrow$ commutativity of NLS flows :

$$\partial_{t_n} u(1) = K_n[u(1)],\ n = 1, 2, \ldots$$

integrability of the original PDE :

$$\frac{d}{ds} K_n[u(1) + sv]|_{s=0} = K_n'[u(1)]v\ ,\ M_n = \partial_{t_n} - K_n'[u(1)]\ ,\ [M_n\ ,\ M_m] = 0$$

$$M_2 u(n-1) + M_3 u(n-2) + \ldots + M_{n-1} u(2) = G_{n+1}\ ,\ n \geq 3$$

$$M_n u(m) = g_n(m)\ ,\ n \geq 2,\ m \geq 2$$

$$\boxed{1}\quad g_n(m) \in \mathcal{P}_{n+m}(m-1)\ ,\quad \boxed{2}\quad M_j g_n(m) = M_n g_j(m)$$

$$g_2(n) + g_3(n-1) + \ldots + g_n(2) = G_{n+2}$$

INTEGRABILITY TEST 1

remark : the existence of $g_2(m) \in \mathcal{P}_{m+2}(m-1)$ and of $g_3(m) \in \mathcal{P}_{m+3}(m-1)$ such that $M_2 g_3(m) = M_3 g_2(m)$ implies the hierarchy $M_n u(m) = g_n(m) \in \mathcal{P}_{m+n}(m-1)$ of commuting flows

$$
\begin{array}{cccc}
g_2(2) \rightarrow & g_3(2)\ , & g_4(2)\ , & g_5(2)\ \ , \\
--- & + & + & + \\
G_4 & g_2(3)\ \ \rightarrow & g_3(3)\ \ , & g_4(3)\ \ , \\
 & --- & + & + \\
 & G_5 & g_2(4)\ \ \rightarrow & g_3(4)\ \ , \\
 & & --- & + \\
 & & G_6 & g_2(5)\ \ \rightarrow \\
 & & & --- \\
 & & & G_7
\end{array}
$$

INTEGRABILITY TEST 2

first obstruction at order $n+1 \;\rightarrow\; A_n-$integrability

$$M_3 g_2(n+1) \notin M_2(\mathcal{P}_{n+4}(n))$$

A_1-integrability $\rightarrow$ NLS
A_2-integrability $\rightarrow M_3 g_2(2) = M_3 G_4$

$$G_4 = (a|u(1)|^2 u_1(1) + bu(1)^2 u_1^*(1)) \in \mathcal{P}_4(1)$$

computation : $M_3(a|u(1)|^2 u_1(1) + bu(1)^2 u_1^*(1)) \in M_2(\mathcal{P}_5(1))$
iff $a = a^* \,,\; b = b^*$

$$\mathcal{P}_3(1) \rightarrow 1 \,,\; \mathcal{P}_4(1) \rightarrow 2 \,,\; \mathcal{P}_5(1) \rightarrow 5 \,,\; \mathcal{P}_6(1) \rightarrow 8 \,,\; \mathcal{P}_4(2) \rightarrow 4$$

$$\mathcal{P}_5(2) \rightarrow 12 \,,\; \mathcal{P}_6(2) \rightarrow 26 \,,\; \mathcal{P}_5(3) \rightarrow 14 \,,\; \mathcal{P}_6(3) \rightarrow 34 \,,\; \mathcal{P}_6(4) \rightarrow 36$$

DP equation 1

testing the 5-parameter family :

$$u_t + cu_x + du_{xxx} - a^2 u_{xxt} - a^2 f(uu_{xxx} + bu_x u_{xx}) + guu_x = 0$$

passing the A_3-integrability test :

1. $a = 0$, KdV , $u_t + cu_x + du_{xxx} + guu_x = 0$
2. $b = 2\,,\ g = 3f$, CH ,
   $u_t + cu_x + du_{xxx} - a^2 u_{xxt} - a^2 f(uu_{xxx} + 2u_x u_{xx}) + 3fuu_x = 0$
3. $b = 3\,,\ g = 4f$, DP ,
   $u_t + cu_x + du_{xxx} - a^2 u_{xxt} - a^2 f(uu_{xxx} + 3u_x u_{xx}) + 4fuu_x = 0$

DP equation 2

DP equation :

$$m_t + um_x + 3u_x m = 0 \ , \quad u - u_{xx} = m$$

Lax pair :

$$\begin{cases} \psi_{xxx} &= \psi_x + \lambda m(x,t)\psi \\ \psi_t &= \frac{1}{\lambda}\psi_{xx} - u(x,t)\psi_x + u_x(x,t)\psi \end{cases}$$

references

Degasperis A, Manakov S V, Santini P M
”Multiple-scale perturbation beyond the Nonlinear Schrödinger Equation.I” Physica D100, 187-211 (1997)

Degasperis A, Procesi M
"Asymptotic integrability", in Degasperis A, Gaeta G, Symmetry and Perturbation Theory, World Scientific, pp. 23 – 37 (1999)

Degasperis A, Holm D D, Hone A N W
"A new integrable equation with peakon solutions", Theoret. and Math. Phys. 133 (2): 1463 – 1474 (2002)

Degasperis A
” Multiscale Expansion and Integrability of Dispersive Wave Equations”, in Mikhailov A V, Integrability, Lect. Notes Phys. 767, Springer (2009)